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SCHUR INDICES OF CHARACTERS OF GROUPS RELATED TO FINITE SPORADIC SIMPLE GROUPS

BY

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ABSTRACT

Let G be a finite sporadic simple group. Then there exist groups $n.G, n.G.2$ and, in case n is even, $n.G.2i$, the group isoclinic to but not isomorphic to $n.G.2$. The Schur indices of all irreducible characters of these groups are computed. In a previous paper this was done for the groups $n.G$ (with one exception). The division algebra corresponding to a character is determined by all the local Schur indices. These are all listed in the tables in Section 6 using the notation from the ATLAS.

1. Introduction

Let G be a finite sporadic simple group. Then G has a cyclic Schur multiplier $\langle z\rangle$ and the outer automorphism group of G has order 1 or 2. In the latter case there exists an outer automorphism σ of order 2 with $\sigma(z) = z^{-1}$. If $|\langle z\rangle| = n$ and σ exists then there exists an "envelope" $n.G.2$ of G. If furthermore n is even, there is also another envelope, the isoclinic group $n.G.2i$. See [2, p. xxiii]. Throughout this paper $n.G.2$ denotes the group whose character table is printed in the ATLAS and $n.G.2i$ denotes the isoclinic group when n is even.

The purpose of this paper is to describe all Schur indices of all irreducible characters of all groups $n.G.2$ and $n.G.2i$. In a previous paper [4] all Schur indices of the groups $n.G$ were computed (there was one ambiguity about a faithful character of $2.Suz$ of degree 228,800). The earlier paper was written before the appearance of the ATLAS [2]. The information in the ATLAS is sufficient to settle the open case mentioned above; see Section 5. Also C. Jansen

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has informed me that he had previously settled this case by using information about the Brauer characters of $2.Suz$. Furthermore, arguments similar to those used in [4], described below, can be used to compute all Schur indices of all envelopes of sporadic groups. The ATLAS, amongst other things, provides a standard notation. Accordingly the tables in Section 6 incorporate those of [4], though the results of [4] are just quoted, not proved again.

As in the proofs in [4], it is important for some of the results to know tables of irreducible Brauer characters. I am grateful to C. Jansen and R.A. Wilson for much information about these [6], [7], [8]. I am also grateful to C. Jansen for computing some inner products.

The notation is standard and is that used in [4].

2. Isoclinism

Throughout this section $M = [M, M]$ is a finite group with a cyclic center of even order and an outer automorphism σ of order 2. Thus M contains a unique central subgroup $\langle z\rangle$ of order 2 and $\sigma(z) = z$.

There exist 2 groups $H = \langle M, y\rangle$ and $\hat{H} = \langle M, \hat{y}\rangle$ with

$$|H\colon M| = |\hat{H}\colon M| = 2$$

such that conjugation by y and $\hat{y}$ induces the automorphism σ on M, and y and $\hat{y}$ can be chosen so that

$$\{y^2, \hat{y}^2\} = \{1, z\}.$$

The groups H and $\hat{H}$ are isoclinic. See [2, p. xxiii].

Let K be a subgroup of M which contains the center of M such that $\sigma(K) = K$. There is a bijection between the conjugacy classes C and $\hat{C}$ of $\langle K, y\rangle - K$ and $\langle K, \hat{y}\rangle - K$ and irreducible characters $\eta, \hat{\eta}$ of $\langle K, y\rangle$ and $\langle K, \hat{y}\rangle$ such that the following holds:

$$(2.1)\qquad \begin{array}{ll} \eta(x) = \hat{\eta}(x) & \text{if } x \in K, \\ \left.\begin{array}{l}\eta(x) = i\hat{\eta}(\hat{x}) \\ \eta(x^2) = -\hat{\eta}(\hat{x}^2)\end{array}\right\} & \text{if } x \in C \subseteq \langle K, y\rangle - K, \quad \hat{x} \in \hat{C} \subseteq \langle K, \hat{y}\rangle - K. \end{array}$$

In particular this applies to the case that $K = M$.

For any finite group K and any irreducible character η of K let

$$\nu_K(\eta) = \frac{1}{|K|}\sum_{x\in K} \eta(x^2).$$

THEOREM 2.2 (Frobenius and Schur): *Let η be an irreducible character of K.*

(i) *If $\eta \neq \bar{\eta}$ then $\nu_K(\eta) = 0$.*

(ii) *If $\eta = \bar{\eta}$ then one of the following occurs:*

 (a) *$m_\infty(\eta) = 1$ and $\nu_K(\eta) = 1$.*

 (b) *$m_\infty(\eta) = 2$ and $\nu_K(\eta) = -1$.*

As a consequence of Theorem 2.2 we prove the following.

THEOREM 2.3: *Let χ be an irreducible character of M.*

(i) *Suppose that $\chi^\sigma = \chi = \bar{\chi}$. Then χ extends to characters ζ, ζ' of H and $\hat{\zeta}, \hat{\zeta}'$ of $\hat{H}$ respectively and the notation can be chosen so that ζ and ζ' are not real valued but $\hat{\zeta}$ and $\hat{\zeta}'$ are real valued with*

$$m_\infty(\hat{\zeta}) = m_\infty(\hat{\zeta}') = m_\infty(\chi).$$

(ii) *Suppose that $\chi^\sigma \neq \chi$. Then the induced characters χ^H and $\chi^{\hat{H}}$ are irreducible. Furthermore*

 (a) *If $\chi = \bar{\chi}$ then $\overline{\chi^\sigma} = \chi^\sigma$ and*

$$\begin{aligned} m_\infty(\chi^H) = m_\infty((\chi^\sigma)^H) = m_\infty(\chi^{\hat{H}}) &= m_\infty((\chi^\sigma)^{\hat{H}}) \\ &= m_\infty(\chi) = m_\infty(\chi^\sigma). \end{aligned}$$

 (b) *If $\chi^\sigma = \bar{\chi} \neq \chi$ then χ^H and $\chi^{\hat{H}}$ are real valued and*

$$m_\infty(\chi^H) \neq m_\infty(\chi^{\hat{H}}).$$

Proof: (i) ζ and ζ' do not vanish on $H - M$. By (2.1) the notation can be chosen so that ζ and ζ' are both not real as

$$\zeta(x) + \zeta'(x) = \chi^H(x) = 0$$

for $x \in H - M$. By Theorem 2.2 this implies that

$$\begin{aligned} 0 = \nu_H(\zeta) &= \frac{1}{|H|}\sum_{x\in M}\zeta(x^2) + \frac{1}{|H|}\sum_{x\in H-M}\zeta(x^2) \\ &= \frac{1}{2}\nu_M(\chi) + \frac{1}{|H|}\sum_{x\in H-M}\zeta(x^2) \end{aligned}$$

and so

$$\frac{1}{|H|}\sum_{x\in H-M}\zeta(x^2) = -\frac{1}{2}\nu_M(\chi).$$

Hence by (2.1)

$$\frac{1}{|H|}\sum_{x\in H-M}\hat{\zeta}(x^2) = -\frac{1}{|H|}\sum_{x\in H-M}\zeta(x^2) = \frac{1}{2}\nu_M(\chi).$$

Therefore

$$\nu_H(\hat{\zeta}) = \frac{1}{|H|}\sum_{x\in M}\chi(x^2) + \frac{1}{2}\nu_M(\chi) = \nu_M(\chi).$$

The result follows from Theorem 2.2.

(ii)(a) If χ is afforded by a real representation, so are χ^σ, χ^H and $\chi^{\hat{H}}$. Conversely if χ^H or $\chi^{\hat{H}}$ is afforded by a real representation so is $\chi + \chi^\sigma$, the restriction to M. As $\chi \neq \chi^\sigma$ both χ and χ^σ are afforded by real representations.

(ii)(b) By Theorem 2.2

$$\sum_{x\in M}\chi^H(x^2) = \sum_{x\in M}\chi^{\hat{H}}(x^2) = 0.$$

Hence

$$\nu_H(\chi^H) = \frac{1}{|H|}\sum_{x\in H-M}\chi^H(x^2),$$
$$\nu_{\hat{H}}(\chi^{\hat{H}}) = \frac{1}{|H|}\sum_{x\in \hat{H}-M}\chi^{\hat{H}}(x^2).$$

By (2.1) this implies that $\nu_H(\chi^H) = -\nu_{\hat{H}}(\chi^{\hat{H}})$. The result follows from Theorem 2.2. ∎

The following rather special result will also be of use in the sequel.

LEMMA 2.4: *Let K be a subgroup of M which contains the center of M such that $\sigma(K) = K$. Let $L = \langle K, y\rangle$ and $\hat{L} = \langle K, \hat{y}\rangle$. Let χ be an irreducible character of M so that χ^H is irreducible and has values in $\mathbb{Q}$. Suppose there exists an irreducible character η of K so that η^L is irreducible with values in $\mathbb{Q}$ and $((\chi^H)_L, \eta^L)$ is odd. Let p be a prime with $p \equiv 3 \pmod 4$. Assume that K has a cyclic S_p-group and the p-block of K which contains η contains an irreducible character η_0 such that η_0^L is reducible. Then one (or both) of the following occurs:*

$$m_p(\chi^H) = m_p(\eta^L) = 2,$$
$$m_p(\chi^{\hat{H}}) = m_p(\eta^{\hat{L}}) = 2.$$

Proof: Since χ^H is rational valued,

$$m_p(\chi^H) = m_p(\eta^L) \leq 2, \quad m_p(\chi^{\hat{H}}) = m_p(\eta^{\hat{L}}) \leq 2.$$

Let $\eta_0^L = \eta_1 + \eta_2$ and let $\eta_0^{\hat{L}} = \hat{\eta}_1 + \hat{\eta}_2$. Then there exists $x \in L - K$ with $\eta_1(x) \neq 0$. Thus the notation can be chosen so that $\hat{\eta}_1(\hat{x}) = i\eta_1(x) \neq 0$. As $p \equiv 3 \pmod 4$, not both η_1 and $\hat{\eta}_1$ have values in $\mathbb{Q}_p$. The result follows from Benard's Theorem, [1] or [4, Theorem 2.12]. ∎

3. Some criteria

In this section H and M are finite groups with $H = \langle M, y\rangle$ and $|H\colon M| = 2$, p is a rational prime. We will prove the main results to be used in the sequel to compute $m_p(\zeta)$ for irreducible characters ζ of H.

THEOREM 3.1: *Let χ be an irreducible character of M and let $\mathbb{Q}(\chi) \subseteq L$ where L is a splitting field of χ.*

(i) *If χ extends to irreducible characters ζ, ζ' of H then $L(\zeta) = L(\zeta')$ is a splitting field of ζ and ζ'.*
(ii) *If χ^H is irreducible then L is a splitting field of χ^H.*

Proof: This follows directly from basic properties of Schur indices. ∎

If, for instance, χ extends to characters ζ and ζ' of H and $m_p(\chi) = 1$, then Theorem 3.1 implies that $m_p(\zeta) = m_p(\zeta') = 1$. However, if χ^H is irreducible and $[\mathbb{Q}_p(\chi)\colon \mathbb{Q}_p(\chi^H)] = 2$, then even if $m_p(\chi) = 1$, there still remains the question of finding $m_p(\chi^H)$.

The following results yield useful criteria for computing $m_p(\zeta)$.

THEOREM A: *Let x be a p'-element in H. Let B be a p-block of H containing the irreducible character ζ. Assume that $\zeta_v(x) \in \mathbb{Q}_p(\zeta)$ for every irreducible character ζ_v in B. Then $m_p(\zeta) \mid \zeta(x)$ in the ring of algebraic integers.*

Proof: This is [4, Corollary 3.2]. ∎

The proofs of the next two results are similar to that of Theorem A. We will say that an algebraic integer is **odd** if it is not divisible by 2.

THEOREM B: *Let B be a p-block of M with $B^y \neq B$. Let x be a p'-element in M so that $\chi_u(x) = \chi_u(x^y)$ for every irreducible character χ_u in B. If $\chi = \chi_v$ is in B with $\chi(x)$ odd then χ^H is irreducible and $m_p(\chi^H)$ is odd.*

Proof: Let $\{\chi_u\}, \{\varphi_i\}$ be the set of all irreducible, Brauer irreducible, characters in B respectively. Since $B \neq B^y$, each χ_u^H, φ_i^H is irreducible. Therefore

$$\chi_u^H = \sum_i d_{ui}\varphi_i^H.$$

Hence

$$\chi_u^H(x) = \sum_i d_{ui}[\mathbb{Q}_p(\chi_u^H, \varphi_i^H) \colon \mathbb{Q}_p(\chi_u^H)]\varphi_i^H(x), \tag{3.2}$$

where the sum ranges over a set of representatives of the algebraic conjugacy classes of irreducible Brauer characters over $\mathbb{Q}_p(\chi_u^H)$.

By assumption

$$\chi_u^H(x) = \chi_u(x) + \chi_u(x^y) = 2\chi_u(x),$$
$$\varphi_i^H(x) = \varphi_i(x) + \varphi_i(x^y) = 2\varphi_i(x),$$

as each φ_i is an integral linear combination of the χ_u restricted to p'-elements. Hence (3.2) implies that

$$\chi_u(x) = \sum_i d_{ui}[\mathbb{Q}_p(\chi_u^H, \varphi_i^H) \colon \mathbb{Q}_p(\chi_u^H)]\varphi_i(x).$$

If $\chi = \chi_u$ so that $\chi_u(x)$ is odd then for some i

$$d_{ui}[\mathbb{Q}_p(\chi_u^H, \varphi_i^H) \colon \mathbb{Q}_p(\chi_u^H)]$$

is odd. Hence $m_p(\chi)$ is odd by [4, Theorem 2.10]. ∎

If, for instance, M has a cyclic center $\langle z \rangle$ with $z^y = z^{-1}$ and B is a faithful block of M, then $B \neq B^y$ in each of the following cases:

$$\begin{array}{ll} |\langle z \rangle| = 3 \text{ or } 6, & p \neq 3, \\ |\langle z \rangle| = 4, & p \neq 2, \\ |\langle z \rangle| = 12, & \text{any } p. \end{array}$$

These are precisely the cases that arise for covering groups of sporadic simple groups with $|\langle z \rangle| > 2$.

LEMMA 3.3: *Suppose that $p \neq 2$. Let φ be an irreducible Brauer character of M. Then either φ^H is irreducible or $\varphi = \varphi^y$ and $\varphi^H = \psi + \psi'$, where $\psi \neq \psi'$ are distinct irreducible Brauer characters of H.*

Proof: Let F be an algebraically closed field of characteristic p and let V be an $F[M]$ module which affords φ. By the Mackey decomposition $(V^H)_M = V \oplus V^y$. Hence if φ^H is reducible, $V^y \simeq V$ and V^H contains an irreducible submodule W such that $W_M = V$. By Higman's Theorem [3, Corollary II.3.10], W is M-projective as $p \neq 2$. Hence $W \mid U^H$ for some indecomposable $F[M]$ module U. As $(U^H)_M = U \oplus U^y$ and $V = W_M \mid (U^H)_M$, this implies that $V \simeq U$ and so $V^H = W \oplus W'$ with W and W' irreducible $F[H]$ modules. Let ψ, ψ' be the Brauer character afforded by W, W' respectively. Then

$$\dim_F(\mathrm{Hom}(W, V^H)) = \dim_F(\mathrm{Hom}(W_M, V)) = 1.$$

Therefore $\psi \neq \psi'$. ∎

THEOREM C: *Assume $p \neq 2$. Let B be a p-block of M and let $\tilde{B}$ be the p-block of H which covers B. Let x be a p'-element of M such that $\chi_u(x^y) = \chi_u(x)$ for all irreducible characters χ_u in B. Suppose that B contains an irreducible character χ with $\chi(x)$ odd and χ^H irreducible. Assume further that every irreducible Brauer character in $\tilde{B}$ has values in $\mathbb{Q}_p$. Then $m_p(\chi^H)$ is odd.*

Proof: Let B_0 be the set of all irreducible Brauer characters φ in B with $\varphi^y = \varphi$. Let B_1 be a set of representatives of the pairs of distinct irreducible Brauer characters $\varphi^y \neq \varphi$ in B. Then

$$\chi = \sum d_i \varphi_i + \sum (d_j \varphi_j + d'_j \varphi_j^y),$$

where the first sum ranges over all $\varphi_i \in B_0$ and the second over all φ_j in B_1. Thus either $d_i \varphi_i(x)$ is odd for some i or $(d_j + d'_j)\varphi_j(x)$ is odd for some j as $\varphi_j^y(x) = \varphi_j(x)$ for all j. By Lemma 3.3 there are 2 extensions $\psi_i \neq \psi'_i$ of every φ_i in B_0. Hence

$$\chi^H = \sum d_i(\psi_i + \psi'_i) + \sum (d_j + d'_j)\varphi_j^H.$$

As d_i is odd for some φ_i in B_0 or $d_j + d'_j$ is odd for some φ_j in B_1 and every ψ_i, ψ'_i and φ_j^H has values in $\mathbb{Q}_p$, the result follows. ∎

4. Outline of method

In this section G is one of the sporadic simple groups which have an outer automorphism of order 2. $M = n.G$ and $H = n.G.2$ or $n.G.2i$ in case n is even.

If χ is an irreducible character of M with $m(\chi) = 1$ and if χ extends to characters ζ, ζ' of H then $m(\zeta) = m(\zeta') = 1$ by Theorem 3.1 and these characters are not mentioned below. Similarly if $m(\chi) = 1$ and χ^H is irreducible with $\mathbb{Q}(\chi^H) = \mathbb{Q}(\chi)$ then $m(\chi^H) = 1$.

In the ATLAS, in case M has a cyclic center of even order, one choice of H is made. In this case $\nu_H(\zeta)$ is given for every irreducible character of ζ of H. Thus $m_\infty(\zeta)$ is determined. By using Theorem 2.3 it is then possible to determine $m_\infty(\hat{\zeta})$ for every irreducible character $\hat{\zeta}$ of $\hat{H}$.

If $p^2 \nmid |G|$, then $p^2 \nmid |H|$ and so $m_p(\zeta)$ can be read off from Benard's Theorem, [1] or [4, Theorem 2.12]. Only minimal information is needed about a p-block B of defect 1 to determine the Schur indices at p of all characters in B. This could be found in the ATLAS. However the procedure is much simplified by using [5].

This leaves m_p for the primes p with $p^2 \mid |G|$.

In Section 5 the sporadic simple groups G which have an outer automorphism of order 2 are listed, followed in each case by the set of all primes p with $p^2 \mid |G|$.

If one of Theorems A, B, or C can be applied, for x in the conjugacy class X of G to a character χ, then χ is listed followed by one or more of Theorem A, X, Theorem B, X, Theorem C, X. In the last case Theorem C, X is followed by a prime for which the criterion is relevant.

For most characters these criteria are sufficient to compute m_p for all but at most one rational prime p. If there is then only one place over p in $\mathbb{Q}(\chi)$, the product formula determines m_p. This is almost always the case.

If the results mentioned above are insufficient, then special arguments are used case by case, as appropriate.

An irreducible Brauer character of degree n is denoted by φ_n.

χ_n^* denotes the faithful character of $n.G.2$ or $n.G.2i$ induced from the character χ_n of $n.G$ in the ATLAS in case the induced character is irreducible.

The final results are listed in Section 6.

5. The groups

$$M_{12}, \{2,3\}$$

All characters of $2.G$ have values in $\mathbb{Q}_3$. Hence $m_3(\zeta) = 1$ for all irreducible characters ζ of $2.G.2$ or $2.G.2i$. Furthermore there is only one place over 2 in every $\mathbb{Q}(\zeta)$.

$$M_{22}, \{2,3\}$$

$$G.2, \qquad 2.G.2, \qquad 2.G.2i.$$

$m(\chi_{10}^*) = m(\chi_{17}^*) = 1$ as $\sqrt{-11} \in \mathbb{Q}_3$.
$\chi_{19}^* = \chi_{20}^*$ is irreducible modulo 3.

$$4.G.2, \qquad 4.G.2i.$$

$\chi_{24}^* = \chi_{25}^*, \chi_{26}^* = \chi_{27}^*, \chi_{28}^* = \chi_{29}^*$ are irreducible modulo 3.
Modulo 3, $\mathbb{Q}(\varphi_{56}^*) = \mathbb{Q}(\sqrt{-2}) \subseteq \mathbb{Q}_3$ and the following hold.
$\chi_{30}^* = \varphi_{56}^* + \varphi_{56}^{'*} + \varphi_{64}^*$.
$\chi_{31}^* = \varphi_{56}^* + \varphi_{56}^{'*} + 2\varphi_{64}^* + \varphi_{160}^* + \varphi_{160}^{'*}$.
There is one place over 2 in each $\mathbb{Q}(\chi_n)$ unless $n = 26$ or 27.

Let x be an element of order 5 in G. Then $\mathbb{N}_{4.G}(\langle x\rangle) = 4.N$ where $N = \mathbb{N}_G(\langle x\rangle)$. Furthermore $|N| = 20$ and there is a unique pair $\eta, \bar\eta$ of faithful irreducible characters of $4.N$. Choose the notation so that $\bar\eta\chi_{26}$ has the center of $4.G$ in its kernel. Let $L = 4.N.2$ or $4.N.2i$. Then $\eta^L = \bar\eta^L$ is rational valued, $\eta(1) = 4, \eta(x) = -1$ and

$$((\chi_{26}^*)_L, \eta^L) = ((\chi_{26}^*)_{4.N}, \eta) = ((\chi_{26})_{4.N}, \eta) = \frac{4.144}{20} + \frac{1}{5} = 29.$$

Similarly $((\chi_{30}^*)_L, \eta^L) = 35$. As χ_{30}^* and η^L are rational valued this implies that $m_p(\chi_{30}^*) = m_p(\eta^L)$ for all p. As $\chi_{26}^* \in \mathbb{Q}_2$ we see that

$$m_2(\chi_{26}^*) = m_2(\eta^L) = m_2(\chi_{30}^*).$$

$$3.G.2$$

$\chi_n^*, n \neq 38$. Theorem B, 1A or 7A. There is only one place over 3 in $\mathbb{Q}(\chi_n^*)$ for all n.

$\mathbb{Q}(\chi_{38}^*) = \mathbb{Q}$, and modulo 3

$\chi_{38}^* = \varphi_{210} + \varphi'_{210}$.

$6.G.2, \qquad 6.G.2i$

$\chi_n^*, n \neq 48, 49, 50$. Theorem B for 5A or 7A. There is only one place over 3 in $\mathbb{Q}(\chi_n^*)$ for all n and there is only one place over 2 in $\mathbb{Q}(\chi_n^*), n = 48, 49, 50$.

Modulo 3

$\chi_{48}^* = \varphi_{210} + \varphi'_{210}$.

$\chi_{49}^*, \chi_{50}^* = \varphi_{308} + \varphi_{56} + \varphi'_{56}$.

$12.G.2, \qquad 12.G.2i$

χ_{53}^*, χ_{54}^*. Theorem B, 7A.

$\chi_n^*, 55 \leq n \leq 59$. Theorem B, 5A.

$J_2, \{2, 3, 5\}$

$G.2$

$\chi_2^* = \chi_3^*, \chi_4^* = \chi_5^*, \chi_8^* = \chi_9^*, \chi_{14}^* = \chi_{15}^*$. Theorem A, 5A.

$\chi_{16}^* = \chi_{17}^*$. Theorem C, 3B for $p = 5$. Theorem A, 15A.

$\mathbb{Q}(\sqrt{6})$ is a splitting field for χ_{21}.

$2.G.2, \qquad 2.G.2i$

By [4] the places with Schur index 2 for $\chi_{24}, \chi_{36}, \chi_{37}, \chi_{38}$ as characters of $2.G$ are 2 and ∞. Since $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\chi_n)$ for $n = 24, 36, 37$ and 38, $m_\infty(\chi_n) = 2$ while $m_p(\chi_n) = 1$ for $p \neq \infty$. Furthermore $\mathbb{Q}(\sqrt{-2})$ splits $\chi_{24}, \chi_{36}, \chi_{37}, \chi_{38}$ by Theorem 3.1.

A similar argument shows that $m_\infty(\chi_n) = 2$ while $m_p(\chi_n) = 1$ for $p \neq \infty$ in case of $2.G.2$ for $n = 31, 34, 35$. Furthermore $\mathbb{Q}(\sqrt{-3})$ splits χ_{31}, χ_{35} and $\mathbb{Q}(\sqrt{-7})$ splits χ_{34}.

$\chi_{22}^* = \chi_{23}^*, \chi_{27}^* = \chi_{28}^*$. Theorem A, 5C.

$\chi_{25}^* = \chi_{26}^*$. Defect 0 for 5.

$\chi_{25}^* = \varphi_{72} + \varphi_{14} + \varphi'_{14}$ modulo 3.

$\chi_{29}^* = \chi_{30}^*$. Theorem A for 15A.

$\chi_{29}^* = \varphi_{100} + \varphi_{14} + \varphi'_{14}$ modulo 3.

$(\chi_{32} - \chi_{33}, \chi_6\chi_{22}) = 1$. Hence $\chi_6\chi_{22} = a\chi_{32} + (a-1)\chi_{33} + \cdots$. Therefore

$$\zeta_6\chi_{22}^* = (2a-1)\chi_{32}^* + \cdots .$$

Hence $m_p(\chi_{32}^*) = 1$ for $p \neq 5$.

$HS, \{2,3,5\}$

All characters of G except $\chi_{11}, \chi_{12}, \chi_{41}, \chi_{42}$ have values in $\mathbb{Q}_5$. Thus $m_5(\zeta) = 1$ for every irreducible character ζ of $2.G.2$ and $2.G.2i$ except possibly $\chi_n^*, n = 11, 12, 41, 42$.

$\chi_{11}^* = \chi_{12}^*$. $m_3(\chi_{11}^*) = 1$ as $\sqrt{-5} \in \mathbb{Q}_3$. Theorem C, 6A for $p = 5$.

$\chi_{14}^* = \chi_{15}^*$. Theorem A, 11A.

$2.G.2, \qquad 2.G.2i$

$\mathbb{Q}(\sqrt{-7})$ splits χ_{32}.

$\mathbb{Q}(\sqrt{-10})$ splits χ_{35}.

$\mathbb{Q}(\sqrt{-15})$ splits χ_{36}.

$\mathbb{Q}(\sqrt{5}, \sqrt{\frac{\sqrt{5}-5}{2}})$ splits χ_{41}, χ_{42}.

$\chi_{26}^* = \chi_{27}^*, \chi_{30}^* = \chi_{31}^*, \chi_{37}^* = \chi_{38}^*$. Irreducible mod 3.

$\chi_{39}^* = \chi_{40}^*$. Theorem A, 11A.

Modulo 5 we have

$\chi_{28}^* = \chi_{29}^* = \varphi_{880} + \varphi_{240} + 2\varphi_{56}$.

$\chi_{33}^* = \chi_{34}^* = \varphi_{912} + \varphi_{1552}$.

$J_3, \{2,3\}$

$\mathbb{Q}(\zeta_9) = \mathbb{Q}(\sqrt{6})$ is a splitting field for ζ_9 and ζ_9'.

Theorem A applies to $\chi_2^* = \chi_3^*$; 19A. $\chi_4^* = \chi_5^*, \chi_7^* = \chi_8^*, \chi_{17}^* = \chi_{18}^*$; 15A. Each of these characters is rational valued.

$\mathbb{Q}(\zeta)$ has only one place over 3 for every irreducible faithful character ζ of $3.G.2$. Theorem B applies to every faithful character with one of the classes 1A, 3A or 17A.

$McL, \{2,3,5\}$

$G.2$

$\mathbb{Q}_2(\chi) = \mathbb{Q}_2(\chi^*)$ for every irreducible character of G. Hence $m_2(\chi^*) = 1$.
$\mathbb{Q}(\sqrt{-5})$ splits χ_{11}. $\mathbb{Q}(\sqrt{-6})$ splits χ_{13}.
Every irreducible Brauer character of $G.2$ has values in $\mathbb{Q}_5$.
$\chi_{16}^* = \chi_{17}^*, \chi_{23}^* = \chi_{24}^*$. Theorem C, 1A for $p = 5$.
$\chi_5^* = \chi_6^*, \chi_{18}^* = \chi_{19}^*$. Theorem C, 3A for $p = 5$.
$\chi_{16}^* = \chi_{17}^*$. Theorem A, 9A.

$3.G.2$

There is exactly one place over 3 in $\mathbb{Q}(\chi^*)$ for every faithful irreducible character $\chi \neq \chi_{25}, \chi_{26}$.

$\chi_{35}^*, \chi_{38}^*, \chi_{39}^*, \chi_{40}^*, \chi_n^*$ for $42 \leq n \leq 45$. Theorem B, 1A.
$\chi_{31}^*, \chi_{32}^*, \chi_{41}^*$. Theorem B, 11A.
χ_{28}^*, χ_{33}^*. Theorem B, 3A.
χ_{29}^*, χ_{30}^*. Theorem A, 7A.
$((\chi_{25} - \chi_{26}), \chi_7\chi_{35}) = 1$. Hence $\chi_7\chi_{35} = a\chi_{25} + (a-1)\chi_{26} + \cdots$ and so $\zeta_7\chi_{35}^* = a\chi_{25}^* + (a-1)\chi_{26}^* + \cdots$. Thus $\mathbb{Q}(\sqrt{-11})$ is a splitting field for χ_{25}^*, χ_{26}^*.
$(\chi_2\chi_{45})^*, \chi_{37}^*) = (\chi_2\chi_{45}, \chi_{37}) = 1$.
$((\chi_3\chi_{45})^*, \chi_{27}^*) = (\chi_3\chi_{45}, \chi_{27}) = 1$.
$((\chi_3\chi_{45})^*, \chi_{36}^*) = (\chi_3\chi_{45}, \chi_{36}) = 21$.
χ_{34}^* is irreducible modulo 5.
$\chi^*_{34|_{3M_{22}}} = \eta_{33}^* + \cdots$.
$m(\eta_{33}^*) = 1$. $\mathbb{Q}(\eta_{33}^*) = \mathbb{Q}(\sqrt{-7}) \subseteq Q_2$. Hence $m_2(\chi_{34}^*) = 1$.

$He, \{2,3,5,7\}$

Theorem A applies to $\chi_2^* = \chi_3^*, \chi_4^* = \chi_5^*, \chi_{17}^* = \chi_{18}^*, \chi_{23}^* = \chi_{24}^*$; 7A. $\chi_7^* = \chi_8^*$; 17A. Each of these characters is rational valued.

Every irreducible Brauer character for $p = 7$ of $G.2$ has values in $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}_7$.

$\chi_{20}^* = \chi_{21}^*$. Theorem A, 21C. Theorem C, 3B for $p = 7$.

$\chi_{10}^* = \chi_{11}^*$. Theorem 1A for $p = 3$ since every Brauer character in the 3-block containing χ_{10}^* is rational valued. χ_{10}^* has defect 0 for $p = 5$. Since $\sqrt{-7} \in \mathbb{Q}_2, m_2(\chi_{10}^*) = 1$.

$Suz, \{2, 3, 5\}$

$2.G$

By Theorem A, 28A, χ_{74} is split at 3. Hence $m_\infty(\chi_{74}) = m_2(\chi_{74}) = 2$.

$G.2$

$\mathbb{Q}(\sqrt{10})$ splits χ_{40}.
$\chi_7^* = \chi_8^*, \chi_{21}^* = \chi_{22}^*$. Theorem A, 9A.
$\chi_{31}^* = \chi_{32}^*$. Theorem A, 13A.
$\chi_{25}^* = \chi_{26}^*$. Theorem A, 21A.
$\chi_{13}^* = \chi_{14}^*$. Theorem A, 15A. Theorem C, 1A for $p = 3$.
$\chi_{18}^* = \chi_{19}^*$. Defect 0 for 5. Theorem C, 1A for $p = 3$.

$2.G.2, \qquad 2.G.2i$

$\mathbb{Q}(\sqrt{-2})$ splits ζ_n for $n = 44, 52, 53, 64, 65, 66, 69, 73, 75$.
$\mathbb{Q}(\sqrt{-5})$ splits ζ_n for $n = 49, 68, 72, 76$.
$\mathbb{Q}(\sqrt{-11})$ splits ζ_{67}.
$\mathbb{Q}(\sqrt{-14})$ splits ζ_{74}.
$\chi_{47}^* = \chi_{48}^*, \chi_{50}^* = \chi_{51}^*$. Theorem A, 9A.
$\chi_{56}^* = \chi_{57}^*$. Theorem A, 21A.
$\chi_{58}^* = \chi_{59}^*$. Theorem A, 15A. Defect 1 for 5.
$\chi_{60}^* = \chi_{61}^*$. Theorem A, 13A.
$(\chi_2\chi_{47}, \chi_{45} - \chi_{46}) = 1 = (\chi_2\chi_{47}, \chi_{55} - \chi_{54})$. Hence

$$\chi_2\chi_{47} = a\chi_{45} + (a-1)\chi_{46} + b\chi_{54} + (b+1)\chi_{55} + \cdots$$

and so

$$\zeta_2\chi_{47}^* = (2a-1)\chi_{45}^* + (2b+1)\chi_{54}^* + \cdots .$$

Therefore $m(\chi_{45}^*) = 2$ or $m(\chi_{54}^*) = 2$ implies that $m(\chi_{47}^*) = 2$ which is not the case.

$(\zeta_2\zeta_{67}, \chi_{62}^*) = (\chi_2\chi_{67}, \chi_{62}) = 1$. ζ_{67} is split by $\mathbb{Q}(\sqrt{-11}), \mathbb{Q}(\sqrt{11}, \sqrt{-11})$ for $2.G.2, 2.G.2i$ respectively. Thus $m_5(\chi_{62}^*) = 1$ and $m_3(\chi_{62}^*) = 1$ in case $2.G.2$.

Let $x \in 2.G$ with its image in G in 12D. Hence $x^{12} = -1$ is in the center of $2.G$. Let x_3 be the 3-part of x and let $S, \hat{S}$ be a S_2-group of $\mathbb{N}_{2.G.2}(\langle x \rangle)$, $\mathbb{N}_{2.G.2i}(\langle x \rangle)$ respectively. Let $L = \langle x \rangle S, \hat{L} = \langle x \rangle \hat{S}$. Thus $|\langle x \rangle| = 24$ and $|L\colon \langle x \rangle| = 8$. Let λ be a faithful linear character of $\langle x \rangle$ and let $\eta = \lambda^L, \hat{\eta} = \lambda^{\hat{L}}$. Then $\eta(1) = 8, \eta(x_3) = -4$ and η vanishes on elements of L distinct from $\pm 1, \pm x_3$.

Let χ be a faithful irreducible character of $2.G$ so that χ^* is irreducible. $\chi^* \eta$ is a character of L with -1 in the kernel. Hence

$$(\chi^*, \eta) = (\chi^* \eta, 1) = \frac{1}{96}(8\chi^*(1) - 8\chi^*(x_3)) = \frac{1}{12}(\chi^*(1) - \chi^*(x_3)).$$

Hence (χ^*_{62}, η) is odd.

Let $p = 3$. The other two irreducible characters η_1 and η_2 in the 3-block of L which contains η are the irreducible constituents of $(\lambda^3)^L$. Now Lemma 2.4 implies that $m_3(\chi^*_{62}) = 2$ for either $2.G.2$ or $2.G.2i$, hence for $2.G.2i$.

Modulo 5

$$\chi^*_{70} = \chi^*_{71} = 2\varphi_{1144} + \varphi_{3784} + \varphi_{122472} + \varphi_{15974}.$$

Hence $m_5(\chi^*_{70}) = 1$.

$\chi^*_{70} = \chi^*_{71}$. Theorem C, 5A for $p = 3$ in case $2.G.2$. Hence $m_3(\chi^*_{70}) = 1$ in this case.

Let $y \in 2.G$ with its image in G in 24A. Let $C = \langle y \rangle \times \langle -1 \rangle$, $L = \mathbb{N}_{2.G.2}(\langle y \rangle)$, $\hat{L} = \mathbb{N}_{2.G.2i}(\langle y \rangle)$. Then $|C| = 48$ and $|L\colon C| = 8$. Let μ be a linear character of C with kernel $\langle -y^{12} \rangle$ and let $\eta = \mu^L, \hat{\eta} = \mu^{\hat{L}}$. Then the following holds:

	$1,$	$y^4,$	$y^8,$	y^{12}
$\eta, \hat{\eta}$	8	4	-4	-8
χ^*_{70}	288288	6	126	336

Furthermore, $\eta(-u) = -\eta(u)$ for $u \in C$ and η vanishes on the remaining elements of C. Hence $\chi^*_{70}\eta$ is a character of L with -1 in its kernel. Thus

$$\begin{aligned}(\chi^*_{70}, \eta) = (\chi^*_{70}\eta, 1) &= \frac{1}{8.24}(8\chi^*_{70}(1) - 8.126 + 8.6 - 8.336) \\ &= \frac{\chi^*_{70}(1)}{24} - 19.\end{aligned}$$

Hence (χ^*_{70}, η) is odd.

Let $p = 3$. The other two irreducible characters η_1, η_2 in the 3-block of L which contains η are the irreducible constituents of $(\mu^3)^L$. Now Lemma 2.4 implies that $m_3(\chi^*_{70}) = 2$ for either $2.G.2$ or $2.G.2i$, hence for $2.G.2i$.

3.G.2

$\mathbb{Q}(\chi)$ has only one place over 3 for every character χ such that $\chi(x)$ is odd for a 3-element x.

$\chi^*_{89}, \chi^*_{107}$. Theorem B, 11A.

χ^*_{108}. Theorem B, 7A.

Since χ^*_{92} and χ^*_{105} are not in the same 3-block as χ^*_{87}, χ^*_{88}, Theorem C may be applied.

χ^*_{92}. Theorem C, 13A for $p = 3$. Defect 0 at 5.

χ^*_{105}. Theorem C, 13A for $p = 3$. Theorem B, 5A.

6.G.2, 6.G.2i

$\chi^*_n, n = 116, 117, 119, 120, 124, 128, 131, 135, 137, 140, 142, 143$. Theorem A, 9A.

$\chi^*_n, n = 115, 118, 138, 141$. Theorem B, 3B.

$\chi^*_n, n = 129, 130, 132, 133, 136$. Theorem B, 11A.

Since χ^*_{132} and χ^*_{133} are in 3-blocks of defect 1, Theorem C can be applied at $p = 3$ for other characters.

$\chi^*_{121}, \chi^*_{122}$. Theorem C, 13A for $p = 3$. Theorem B, 5A.

χ^*_{126}. Theorem C, 13A for $p = 3$. Defect 0 for 5.

χ^*_{134}. Theorem C, 5A for $p = 3$. Defect 0 for 5.

$\chi^*_{123} = \zeta_5 \chi^*_{115}$.

$$(\zeta_4 \chi^*_{132}, \chi^*_{125}) = (\chi_4 \chi_{132}, \chi_{125}) = 1.$$
$$(\zeta_6 \chi^*_{132}, \chi^*_{127}) = (\chi_6 \chi_{132}, \chi_{127}) = 13.$$

So $\mathbb{Q}(\sqrt{-39})$ splits χ^*_{125} and χ^*_{127}. Hence

$$m_2(\chi^*_{125}) = m_5(\chi^*_{125}) = m_2(\chi^*_{127}) = m_5(\chi^*_{127}) = 1.$$
$$(\zeta_{43} \chi^*_{115}, \chi^*_{139}) = (\chi_{43} \chi_{115}, \chi_{139}) = 1.$$

Hence $m_p(\chi^*_{139}) = 1$ for $p \neq 3$.

$ON, \{2,3,7\}$

$\mathbb{Q}_7(\chi) = \mathbb{Q}_7(\chi^*)$ for every irreducible character χ of $3.G$. Hence $m_7(\chi^*) = 1$ for all χ.

$\chi_3^* = \chi_4^*$. Theorem A, 31A.

$\chi_5^* = \chi_6^*$. $\sqrt{5} \in \mathbb{Q}_3$ and so $m_3(\chi_5^*) = 1$.

$\chi_{16}^* = \chi_{17}^*$. Modulo 3, φ_1 has multiplicity 3 in χ_{16}, and so in χ_{16}^*.

If χ is an irreducible faithful character of $3.G.2$ with $\chi(1)$ odd or $\chi(x)$ odd for x in 5A, then $m_2(\chi^*) = 1$. In each of these cases there is only one place over 3 in $\mathbb{Q}(\chi^*)$ unless $\chi = \chi_{38}$ or χ_{39}.

Modulo 5 χ_{38}^* and χ_{39}^* contain φ_{51522} with multiplicity 1.

For the remaining characters χ, there is only one place over 3 in $\mathbb{Q}(\chi)$.

Theorem B for 19A can be applied to a cubic extension of $\mathbb{Q}$ to handle χ_{45}^* and χ_n^* for $50 \le n \le 55$.

χ_{46}^*, χ_{47}^*. Theorem B, 7A.

$Fi_{22}, \{2,3,5\}$

$\chi_{22}^* = \chi_{23}^*$. Theorem A, 13A.

$\chi_{40}^* = \chi_{41}^*, \chi_{51}^* = \chi_{52}^*$. Theorem A, 11A.

$\chi_{33}^* = \chi_{34}^*$. Defect 0 for 5. $m_3(\chi_{33}^*) = 1$ as $\sqrt{-2} \in \mathbb{Q}_3$.

$\chi_{43}^* = \chi_{44}^*$. $(\zeta_{45}\zeta_{48}, \chi_{43}^*) = (\chi_{45}\chi_{48}, \chi_{43}) = 5211$.

$2.G.2, \qquad 2.G.2i$

$\chi_{69}^* = \chi_{70}^*, \chi_{71}^* = \chi_{72}^*$. Theorem A, 11A.

$\chi_{105}^* = \chi_{106}^*$. Theorem A, 13A.

$3.G.2$

All characters with $\chi(x)$ odd for x of order 1, 3 or 9 are settled by Theorem B.

χ_{140}^*. Theorem B, 21A.

χ_{154}^*. Theorem B, 5A. Defect 1 for 5.

$\chi_{148}^*, \chi_{149}^*$. $(\chi_{148}, \chi_{65}\chi_{140}) = (\chi_{149}, \chi_{65}\chi_{140}) = 33283$.

Hence $\zeta_{65}\chi_{140}^* = 33283(\chi_{148}^* + \chi_{149}^*) + \cdots$.

$6G2, \qquad 6G2i$

All characters except χ_{194} are settled by Theorem B and one of 7A, 9A, 21A or Theorem A, 13A.

$$(\chi^*_{194}, \chi_2\chi^*_{173}) = (\chi_{194}, \chi_2\chi_{173}) = 1.$$

Thus $m_p(\chi^*_{194}) = m_p(\chi^*_{173})$ for all primes p.

$HN, \{2,3,5\}$

$\mathbb{Q}(\sqrt{6})$ is a splitting field for χ_{42}.
$\chi^*_2 = \chi^*_3, \chi^*_6 = \chi^*_7, \chi^*_{13} = \chi^*_{14}, \chi^*_{27} = \chi^*_{28}$. Theorem A, 5C.
$\chi^*_{11} = \chi^*_{12}, \chi^*_{21} = \chi^*_{22}, \chi^*_{30} = \chi^*_{31}$. Theorem A, 25A.
$\chi^*_{25} = \chi^*_{26}$. Theorem A, 19A.
$\chi^*_{15} = \chi^*_{16}$. Theorem A, 35A.
$(\zeta^2_{34}, \chi^*_{35}) = (\chi^2_{34}, \chi_{35}) = 9223$. Hence $m\ (\chi_{35}) = 1$.

$Fi'_{24}, \{2,3,5,7\}$

Every character of $G.2$ lies in $\mathbb{Q}(\sqrt{33}, \sqrt{-23}, \sqrt{13})$ and so every Brauer character modulo 3 lies in $\mathbb{Q}(\sqrt{-23}, \sqrt{13}) \subseteq \mathbb{Q}_3$. Every character in a 5-block of positive defect lies in $\mathbb{Q} \subseteq \mathbb{Q}_5$.

$\chi^*_{14} = \chi^*_{15}$. $\sqrt{-3} \in \mathbb{Q}_7$, defect 0 for 5, Theorem C, 7A for $p = 3$. Thus $m_p(\chi^*_{14}) = 1$ for $p = 3, 5$ or 7.

$\chi^*_{60} = \chi^*_{61}$, $\chi^*_{77} = \chi^*_{78}$. Theorem A, 21A. Theorem C for $p = 3$ with 17A for χ^*_{60} and 5A for χ^*_{77}.

$\chi^*_{69} = \chi^*_{70}$. Theorem A, 39C.
$\chi^*_{91} = \chi^*_{92}$. Theorem A, 29A.
$\chi^*_{99} = \chi^*_{100}$. Theorem A, 45A. Theorem C, 3E for $p = 5$.
$\chi^*_{101} = \chi^*_{102}$. Defect 0 for 5 and 7. Theorem C, 13A for $p = 3$.
$\chi^*_{128}, \chi^*_{129}, \chi^*_{171}, \chi^*_{173}$. Theorem B, 13A.
$\chi^*_{172}, \chi^*_{175}$. Theorem B, 11A.
χ^*_{135}. Theorem B, 5A and 7A.
χ^*_{174}. Theorem B, 33A.

6. Tables

The first column gives the name of the group.

If n is even $n.G.2$ denotes the group whose character table is printed in the ATLAS and $n.G.2i$ denotes the group isoclinic to, but not isomorphic to, $n.G.2$.

The second column identifies the character.

n^* denotes the faithful character of $H.2$ or $H.2i$ induced from the character χ_n of H in the ATLAS in case the induced character is irreducible. n, n' denote the 2 extensions of χ_n to $H.2$ or $H.2i$ otherwise.

In case $G = M_{22}$, $\overline{30}$ denotes the complex conjugate of 30 for $4.G$. For $12.G.2$ α is the automorphism of $\mathbb{Q}(\sqrt{11}, \sqrt{3})$ whose fixed field is $\mathbb{Q}(\sqrt{11})$ and β is the nonidentity automorphism of $\mathbb{Q}(\sqrt{3})$.

The third column contains the degree of the character.

The fourth column has the field generated by the character values.

The last column has a list of all the places in $\mathbb{Q}$ for which the Schur index is not 1, hence 2. Characters with Schur index 1 are not listed.

M_{12}

$2.G$	19	32	$\mathbb{Q}$	∞,2
$2.G.2$	$20^* = 21^*$	88	$\mathbb{Q}$	2,5
$2.G.2i$	$19, 19'$	32	$\mathbb{Q}(\sqrt{2})$	∞
	$20^* = 21^*$	88	$\mathbb{Q}$	∞,5
	$22^* = 23^*$	220	$\mathbb{Q}$	∞,2
	$25^* = 26^*$	320	$\mathbb{Q}$	∞,11

M_{22}

$2.G.2i$	$17^* = 18^*$	252	$\mathbb{Q}$	∞,11
	$19^* = 20^*$	308	$\mathbb{Q}$	∞,2
$4.G$	$30, \overline{30}$	176	$\mathbb{Q}(\sqrt{-1})$	5
$4.G.2$	$28^*, 29^*$	320	$\mathbb{Q}(\sqrt{11})$	∞
	30^*	352	$\mathbb{Q}$	∞,5
	31^*	1120	$\mathbb{Q}$	∞,2
$4.G.2i$	$26^*, 27^*$	288	$\mathbb{Q}(\sqrt{-7})$	2
	30^*	352	$\mathbb{Q}$	2,5
$6.G.2$	52^*	768	$\mathbb{Q}$	3,5

$6.G.2i$	45^*	240	$\mathbb{Q}$	∞,3
	$46^*, 47^*$	252	$\mathbb{Q}(\sqrt{33})$	∞
	48^*	420	$\mathbb{Q}$	∞,2
	$49^*, 50^*$	420	$\mathbb{Q}(\sqrt{3})$	∞
	51^*	660	$\mathbb{Q}$	∞,3
	52^*	768	$\mathbb{Q}$	∞,5
$12.G.2$	$57^*, 57^{*\alpha}, 58^*, 58^{*\alpha}$	672	$\mathbb{Q}(\sqrt{11}, \sqrt{3})$	∞
	$59^*, 59^{*\beta}$	768	$\mathbb{Q}(\sqrt{3})$	∞
	J_2			
G	21	336	$\mathbb{Q}$	2,3
$2.G$	22,23	6	$\mathbb{Q}(\sqrt{5})$	∞
	24	14	$\mathbb{Q}$	∞,2
	27,28	56	$\mathbb{Q}(\sqrt{5})$	∞
	29,30	64	$\mathbb{Q}(\sqrt{5})$	∞
	31	84	$\mathbb{Q}$	∞,3
	32,33	126	$\mathbb{Q}(\sqrt{5})$	∞
	34	216	$\mathbb{Q}$	∞,7
	35	252	$\mathbb{Q}$	∞,3
	36	336	$\mathbb{Q}$	∞,2
	37	350	$\mathbb{Q}$	∞,2
	38	448	$\mathbb{Q}$	∞,2
$2.G.2$	$22^* = 23^*$	12	$\mathbb{Q}$	∞,5
	$24, 24'$	14	$\mathbb{Q}(\sqrt{2})$	∞
	$25^* = 26^*$	100	$\mathbb{Q}$	∞,2
	$27^* = 28^*$	112	$\mathbb{Q}$	∞,5
	$29^* = 30^*$	128	$\mathbb{Q}$	∞,5
	$31, 31'$	84	$\mathbb{Q}(\sqrt{3})$	∞
	$32^* = 33^*$	252	$\mathbb{Q}$	∞,5
	$34, 34'$	216	$\mathbb{Q}(\sqrt{7})$	∞
	$35, 35'$	252	$\mathbb{Q}(\sqrt{3})$	∞
	$36, 36'$	336	$\mathbb{Q}(\sqrt{2})$	∞
	$37, 37'$	350	$\mathbb{Q}(\sqrt{2})$	∞
	$38, 38'$	448	$\mathbb{Q}(\sqrt{2})$	∞
$2.G.2i$	$22^* = 23^*$	12	$\mathbb{Q}$	∞,5
	$27^* = 28^*$	112	$\mathbb{Q}$	∞,5
	$29^* = 30^*$	128	$\mathbb{Q}$	∞,5
	$32^* = 33^*$	252	$\mathbb{Q}$	∞,5

HS

$2.G$	32	1000	$\mathbb{Q}$	∞,7
	35	1792	$\mathbb{Q}$	∞,2
	36	1848	$\mathbb{Q}$	∞,3
	41,42	2520	$\mathbb{Q}(\sqrt{5})$	∞
$2.G.2i$	$26^* = 27^*$	352	$\mathbb{Q}$	∞,2
	$28^* = 29^*$	1232	$\mathbb{Q}$	∞,2
	$30^* = 31^*$	1848	$\mathbb{Q}$	∞,2
	$32, 32'$	1000	$\mathbb{Q}(\sqrt{7})$	∞
	$33^* = 34^*$	2464	$\mathbb{Q}$	∞,2
	$35, 35'$	1792	$\mathbb{Q}(\sqrt{10})$	∞
	$36, 36'$	1848	$\mathbb{Q}(\sqrt{15})$	∞
	$37^* = 38^*$	3960	$\mathbb{Q}$	∞,2
	$39^* = 40^*$	4608	$\mathbb{Q}$	∞, 11
	$41, 41', 42, 42'$	2520	$\mathbb{Q}(\sqrt{\frac{5-\sqrt{5}}{2}})$	∞

J_3

G	9	816	$\mathbb{Q}$	2,3

McL

G	11	3520	$\mathbb{Q}$	∞, 5
	13	4752	$\mathbb{Q}$	∞, 3

Ru

$2.G$	49,50	8192	$\mathbb{Q}(\sqrt{29})$	∞
	53	34944	$\mathbb{Q}$	∞, 3
	56	48256	$\mathbb{Q}$	∞, 2
	57	87696	$\mathbb{Q}$	∞, 2
	60	221184	$\mathbb{Q}$	∞, 2
	61	250560	$\mathbb{Q}$	∞, 2

Suz

G	40	197120	$\mathbb{Q}$	2,5
$2.G$	44	220	$\mathbb{Q}$	$\infty, 2$
	49	4928	$\mathbb{Q}$	$\infty, 5$
	52	20020	$\mathbb{Q}$	$\infty, 2$
	53	20020	$\mathbb{Q}$	$\infty, 2$
	56,57	35100	$\mathbb{Q}(\sqrt{21})$	∞
	58,59	60060	$\mathbb{Q}(\sqrt{5})$	∞
	60,61	61236	$\mathbb{Q}(\sqrt{13})$	∞
	62,63	79872	$\mathbb{Q}(\sqrt{2})$	∞
	64	80080	$\mathbb{Q}$	∞ ,2
	65	80080	$\mathbb{Q}$	∞, 2
	66	100100	$\mathbb{Q}$	∞ ,2
	67	102400	$\mathbb{Q}$	∞ ,11
	68	128128	$\mathbb{Q}$	∞ ,5
	69	137280	$\mathbb{Q}$	∞ ,2
	70,71	144144	$\mathbb{Q}(\sqrt{3})$	∞
	72	192192	$\mathbb{Q}$	∞ ,5
	73	197120	$\mathbb{Q}$	∞ ,2
	74	228800	$\mathbb{Q}$	∞ ,2
	75	277200	$\mathbb{Q}$	∞ ,2
	76	315392	$\mathbb{Q}$	∞ ,5
$2.G.2$	$54^* = 55^*$	70200	$\mathbb{Q}$	3,7
	$56^* = 57^*$	70200	$\mathbb{Q}$	∞ ,3
	$58^* = 59^*$	120120	$\mathbb{Q}$	∞ ,5
	$60^* = 61^*$	122472	$\mathbb{Q}$	∞ ,13
	$62^* = 63^*$	159744	$\mathbb{Q}$	∞ ,2
	$70^* = 71^*$	288288	$\mathbb{Q}$	∞ ,2
$2.G.2i$	$44, 44'$	220	$\mathbb{Q}(\sqrt{2})$	∞
	$45^* = 46^*$	728	$\mathbb{Q}$	∞ ,3
	$47^* = 48^*$	1144	$\mathbb{Q}$	∞ ,3
	$49, 49'$	4928	$\mathbb{Q}(\sqrt{5})$	∞
	$50^* = 51^*$	32032	$\mathbb{Q}$	∞ ,3
	$52, 52'$	20020	$\mathbb{Q}(\sqrt{2})$	∞
	$53, 53'$	20020	$\mathbb{Q}(\sqrt{2})$	∞
	$54^* = 55^*$	70200	$\mathbb{Q}$	∞ ,3
	$56^* = 57^*$	70200	$\mathbb{Q}$	∞ ,3
	$58^* = 59^*$	120120	$\mathbb{Q}$	∞ ,5
	$60^* = 61^*$	122472	$\mathbb{Q}$	∞ ,13
	$62^* = 63^*$	159744	$\mathbb{Q}$	∞ ,3
	$64, 64'$	80080	$\mathbb{Q}(\sqrt{2})$	∞
	$65, 65'$	80080	$\mathbb{Q}(\sqrt{2})$	∞
	$66, 66'$	100100	$\mathbb{Q}(\sqrt{2})$	∞
	$67, 67'$	102400	$\mathbb{Q}(\sqrt{11})$	∞
	$68, 68'$	128128	$\mathbb{Q}(\sqrt{5})$	∞

	69, 69′	137280	$\mathbb{Q}(\sqrt{2})$	∞
	70* = 71*	288288	$\mathbb{Q}$	∞ ,3
	72, 72′	192192	$\mathbb{Q}(\sqrt{5})$	∞
	73, 73′	197120	$\mathbb{Q}(\sqrt{2})$	∞
	74, 74′	228800	$\mathbb{Q}(\sqrt{14})$	∞
	75, 75′	277200	$\mathbb{Q}(\sqrt{2})$	∞
	76, 76′	315392	$\mathbb{Q}(\sqrt{5})$	∞
$6.G.2i$	115*	24	$\mathbb{Q}$	∞ ,3
	116*	1560	$\mathbb{Q}$	∞ ,3
	117*	1848	$\mathbb{Q}$	∞ ,3
	118*	8736	$\mathbb{Q}$	∞ ,3
	119*	8736	$\mathbb{Q}$	∞ ,3
	120*	17160	$\mathbb{Q}$	∞ ,3
	123*	24024	$\mathbb{Q}$	∞ ,3
	124*	24024	$\mathbb{Q}$	∞ ,3
	125*	24024	$\mathbb{Q}$	∞ ,3
	126*	46200	$\mathbb{Q}$	∞ ,3
	127*	54912	$\mathbb{Q}$	∞ ,3
	128*	54912	$\mathbb{Q}$	∞ ,3
	131*	85800	$\mathbb{Q}$	∞ ,3
	134*	154440	$\mathbb{Q}$	∞ ,3
	135*	211200	$\mathbb{Q}$	∞ ,3
	136*	224640	$\mathbb{Q}$	∞ ,3
	137*	279552	$\mathbb{Q}$	∞ ,3
	138*	288288	$\mathbb{Q}$	∞ ,3
	139*	288288	$\mathbb{Q}$	∞ ,3
	140*	343200	$\mathbb{Q}$	∞ ,3
	141*	360360	$\mathbb{Q}$	∞ ,3
	142*	600600	$\mathbb{Q}$	∞ ,3
	143*	873600	$\mathbb{Q}$	∞ ,3

ON

$G.2$	5* = 6*	51832	$\mathbb{Q}$	2,5
$3.G.2$	44*	233244	$\mathbb{Q}$	3,5

Fi_{22}

$6.G.2$	173*	247104	$\mathbb{Q}$	∞ ,3
	194*	5189184	$\mathbb{Q}$	∞ ,3

HN

G	42	2661120	$\mathbb{Q}$	2,3

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